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Suzuki Type Common Fixed Point Theorem in Complete Metric Space and Partial Metric Space

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Abstract. In this paper we establish that a pair of compatible mappings have unique common fixed point in metric and partial metric spaces respectively. The mappings are Suzuki type. We give examples to illustrate our results.

1. Introduction

Meir-Keeler[19] made a new type generalizations of Banach contraction principle[5] in 1969.The Banach contraction principle[5] plays an important role in fixed point theory. Fixed point theory is an important and powerful tool to study the phenomenon of nonlinear analysis. Banach contraction principle has many generalizations in various branches of mathematics. Some of these generalization in metric spaces are in [1, 6–9, 16, 19, 20, 30].
Partial metric spaces are spaces where we have the concept of non-zero self distances. The motivation behind this concept was to obtain a modified version of Banach contraction principle, more generally to solve certain problems arising in computer science. The need for such study arose in computer science where a metric approach to certain problems of denotational semantics[31] can be modified to incorporate non-zero self-distances. Elementary fixed point results having important implications in computer sciences was proved in the introductory papers[17, 18]. After that a number of papers on fixed points have appeared, some references being [2–4, 11, 13, 21, 26, 32].
Recently, Suzuki [19] proved two fixed point theorems, one of which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness. The Banach contraction does not have this property. Another one is a generalization of Meir-Keeler's result. The work of Suzuki[28] also provides with a new methodology of proof which has been followed afterwards in a number of papers [14, 15, 22, 23, 29, 33]. There are also direct generalization of this result in works like [24].
The concept of compatible mappings was introduced by Jungck[12] as a generalization of commuting mappings. The utility of compatibility in the context of fixed point theory was demonstrated by extending a theorem of Park-Bae [25]. Recently Samet et al[27] introduced a definition of compatible pair of mappings in partial metric spaces.
In this paper we prove that two compatible mapping have unique common fixed point in metric and partial metric spaces respectively. The result is supported with examples. In the corresponding metric spaces, the result generalizes a theorem of[23].

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2. Preliminaries

Definition 2.1. *[17] Let X be a nonempty set and let $p : X \times X \to \mathbb{R}^+$ be such that the following are satisfied. For all $x, y, z \in X$,*
(P_1) $x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y)$;
(P_2) $p(x, x) \leq p(x, y)$;
(P_3) $p(x, y) = p(y, x)$;
(P_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
Then the pair (X, p) is called a partial metric space and p is called a partial metric on X.

It is clear that if $p(x, y) = 0$, then from (P_1) and (P_2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0. If p be a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$ defined as

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

satisfies the conditions of an usual metric on X. Each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where

$B_p(x, \varepsilon) = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

The concepts of convergence, Cauchy sequence, completeness and continuity in partial metric space are given in the following definition.

Definition 2.2. *[17] Let (X, p) be a partial metric space.*
(1) A sequence $\{x_n\}$ in the partial metric space (X, p) converges to the limit x if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.
(2) A sequence$\{x_n\}$ in the partial metric space (X, p) is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and is finite.
(3)A partial metric space (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = \lim_{m,n\to\infty} p(x_m, x_n)$.
(4) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subseteq B_p(f(x_0), \varepsilon)$.

The definition of continuity described above is equivalent to the following statement.
A function $f : X \to X$, where (X, p) is a partial metric space, is continuous if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$ as $n \to \infty$.

Lemma 2.3. *[17] Let (X, p) be a partial metric space.*
(1) A sequence $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).
(2) A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover $\lim_{n\to\infty} d_p(x, x_n) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{m,n\to\infty} p(x_m, x_n)$.

Definition 2.4. *[11] A sequence $\{x_n\}$ in a partial metric space $(X,\ p)$ is called 0-Cauchy if $\lim\limits_{n,\ m\to\infty} p(x_n,\ x_m) = 0$. We say that $(X,\ p)$ is 0-complete if each 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x,\ x) = 0$.*

Note that, each 0-Cauchy sequence in $(X,\ p)$ is Cauchy in $(X,\ d_p)$ and every complete partial metric space is 0-complete.

Proposition 2.5. *[10] Let $(X,\ p)$ be a partial metric space. Then the function $d : X \times X \longrightarrow [0,\ \infty)$ defined by $d(x,\ y) = 0$ whenever $x = y$ and $d(x,\ y) = p(x,\ y)$ whenever $x \neq y$, is a metric on X such that $\tau_{d_p} \subseteq \tau_d$. Moreover, $(X,\ d)$ is complete if and only if $(X,\ p)$ is 0-complete.*

Theorem 2.6. *[28] Define a function θ from $[0,1)$ onto $(\frac{1}{2},1]$ by*

$$\theta(r)=\begin{cases}1, & \text{if} \quad 0\leq r\leq \frac{(\sqrt{5}-1)}{2};\\ \frac{1-r}{r^2}, & \text{if} \quad \frac{(\sqrt{5}-1)}{2}\leq r\leq 2^{-\frac{1}{2}};\\ \frac{1}{1+r}, & \text{if} \quad \frac{1}{\sqrt{2}}\leq r<1.\end{cases}$$

Let (X,d) be a complete metric space. T is a mapping on X. If T satisfy the following

$$\theta(r)d(x,Tx)\leq d(x,y)\Rightarrow d(Tx,Ty)\leq rd(x,y) \text{ for all } x,y\in X,$$

then T has a fixed point.

Definition 2.7. *[12] Let S and T be mappings from a metric space(X,d) into itself. Then S and T are said to be compatible if $lim_{n\to\infty}d(STx_n,TSx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty}Sx_n=lim_{n\to\infty}Tx_n=z$ for some z in X. Thus, if $d(STx_n,TSx_n)\to 0$ as $d(Sx_n,Tx_n)\to o$, then S and T are compatible.*

Definition 2.8. *[27] Let (X,p) be a partial metric space and $F,g:X\to X$ are mappings of X into itself. We say that the pair $\{F,g\}$ is partial compatible if the following conditions hold:*
(b_1) $p(x,x)=0\Rightarrow p(gx,gx)=0$;
(b_2) $\lim_{n\to\infty}p(Fgx_n,gFx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $Fx_n\to t$ and $gx_n\to t$ for some $t\in X$.

3. Main Result

Result in metric space

Theorem 3.1. *Let (X,d) be a complete metric space. Let S be a continuous mappings on X and T be another mapping on X such that $\{T,S\}$ is compatible and $T(X)\subset S(X)$. Also let for all $x,y\in X$ and for any $\varepsilon>0$ there exist $\delta(\varepsilon)>0$ such that*

$$(i)\frac{1}{2}d(Sx,Tx)<d(Sx,Sy)\Rightarrow d(Tx,Ty)<\max\{d(Sx,Sy),\frac{1}{2}(d(Sx,Tx)+d(Sy,Ty))\}; \tag{1}$$

$$(ii)\frac{1}{2}d(Sx,Tx)<d(Sx,Sy) \text{ and } \max\{d(Sx,Sy),\frac{1}{2}(d(Sx,Tx)+d(Sy,Ty))\}<\varepsilon+\delta(\varepsilon)$$
$$\Rightarrow d(Tx,Ty)\leq\varepsilon. \tag{2}$$

Then there exists a unique common fixed point of S and T.

Proof. Since $T(X)\subset S(X)$, therefore for any $x\in X$ there exists $y\in X$ such that $Tx=Sy$.
So we can define a mapping I on X satisfying $SIx=Tx$ for all $x\in X$. Thus it is clear that,

$$Ix\begin{cases}=x, & \text{if} \quad Sx=Tx;\\ \neq x, & \text{if} \quad Sx\neq Tx.\end{cases}$$

For $x\in X$ with $Sx\neq Tx$, we have

$$d(Sx,Tx)<2d(Sx,Tx)=2d(Sx,SIx).$$

It follows from (1) that

$$\begin{aligned}d(Tx,TIx)&<\max\{d(Sx,SIx),\frac{1}{2}(d(Sx,Tx)+d(SIx,TIx))\}\\&\leq\max\{d(Sx,Tx),d(Tx,TIx)\}\end{aligned}$$

therefore,

$$d(SIx, SIIx) < d(Sx, SIx) \tag{3}$$

for all $x \in X$ with $Sx \neq SIx$.
For $x \in X$ with $Sx = Tx$, we have $Ix = x$, so

$$d(SIx, SIIx) \leq d(Sx, SIx) \text{ for all } x \in X. \tag{4}$$

Let $u \in X$. Put $u_0 = u$ and $u_n = I^n u$ for all $n \in \mathbb{N}$. By (4), $\{d(Su_n, Su_{n+1})\}$ is a real decreasing sequence of positive terms and hence converges to some $\alpha \geq 0$. Suppose $\alpha > 0$, then by (3), $\{d(Su_n, Su_{n+1})\}$ is strictly decreasing and hence $d(Su_n, Su_{n+1}) > \alpha$ for all $n \in \mathbb{N}$.
Take $j \in \mathbb{N}$ with $d(Su_j, Su_{j+1}) < \alpha + \delta(\alpha)$.
Since

$$\begin{aligned}
&\max\{d(Su_j, Su_{j+1}), \frac{1}{2}(d(Su_j, Tu_j)) + d(Su_{j+1}, Tu_{j+1})\} \\
&\qquad = \max\{d(Su_j, Su_{j+1}), \frac{1}{2}(d(Su_j, Su_{j+1})) + d(Su_{j+1}, Su_{j+2})\} \\
&\qquad = d(Su_j, Su_{j+1})
\end{aligned}$$

it follows by (2) that $d(Tu_j, Tu_{j+1}) = d(Su_{j+1}, Su_{j+2}) \leq \alpha$. This is a contradiction. Therefore $\alpha = 0$.
So,

$$\lim_{n\to\infty} d(Su_n, Su_{n+1}) = 0. \tag{5}$$

Fix $\varepsilon > 0$ and put $\delta_1 = \min\{\varepsilon, \delta(\varepsilon)\}$. By (5) we can choose $\nu_1 \in \mathbb{N}$ such that

$$d(Su_n, Su_{n+1}) < \delta_1 \text{ for all } n \geq \nu_1.$$

Fix $l \in \mathbb{N}$ with $l \geq \nu_1$.
We shall show, by induction, that

$$d(Su_l, Su_{l+m}) < \varepsilon + \delta_1 \tag{6}$$

for all $m \in \mathbb{N}$.
If $m = 1$, (6) is obvious. Suppose that $d(Su_l, Su_{l+m}) < \varepsilon + \delta_1$ holds for some $m \in \mathbb{N}$.
When $d(Su_l, Su_{l+m}) \leq \varepsilon$ we have,

$$\begin{aligned}
p(Su_l, Su_{l+m+1}) &\leq d(Su_l, Su_{l+m}) + d(Su_{l+m}, Su_{l+m+1}) \\
&< \varepsilon + \delta_1.
\end{aligned}$$

In the other case, when $\varepsilon < d(Su_l, Su_{l+m}) < \varepsilon + \delta_1$, we have

$$d(Su_l, Tu_l) = d(Su_l, Su_{l+1}) < \delta_1 \leq \varepsilon < d(Su_l, Su_{l+m}) \leq 2d(Su_l, Su_{l+m}).$$

Therefore, $\frac{1}{2}d(Su_l, Tu_l) \leq d(Su_l, Su_{l+m})$.
Moreover,

$$\begin{aligned}
\max\{d(Su_l, Su_{l+m}), \frac{1}{2}(d(Su_l, Su_{l+1}) + d(Su_{l+m}, Su_{l+m+1}))\} &< \max\{\varepsilon + \delta_1, \frac{\delta_1 + \delta_1}{2}\} \\
&= \varepsilon + \delta_1 \\
&\leq \varepsilon + \delta(\varepsilon).
\end{aligned}$$

By (2) we obtain

$$d(Su_{l+1}, Su_{l+m+1}) = d(Tu_l, Tu_{l+m}) \leq \varepsilon$$

Hence,

$$\begin{aligned} d(Su_l, Su_{l+m+1}) &\leq d(Su_l, Su_{l+1}) + d(Su_{l+1}, Su_{l+m+1}) \\ &\leq \varepsilon + \delta_1 \end{aligned}$$

so by induction (6) holds for all $m \in \mathbb{N}$.
Since ε is arbitrary, we have

$$\limsup_{n\to\infty,\ m>n} d(Su_m, Su_n) < \infty.$$

Therefore $\{Su_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete there exist $z \in X$ such that

$$Su_n \to z. \text{ So } Tu_n = Su_{n+1} \to z. \tag{7}$$

Next we show that z is a fixed point of S. Arguing by contradiction, we assume that $Sz \neq z$. Also we denote by $\beta = d(Sz, z)$. Obviously, we have $\beta > 0$.
Therefore we have,

$$d(Su_n, Su_{n+1}) \to 0, \quad d(SSu_n, SSu_{n+1}) \to 0, \quad d(Su_n, SSu_n) \to \beta,$$

Therefore, $\nu_2 \in \mathbb{N}$ such that

$$d(Su_n, Su_{n+1}) < \frac{\beta}{2}, \quad d(SSu_n, SSu_{n+1}) < \frac{\beta}{2}, \quad d(Su_n, SSu_n) > \frac{\beta}{2} \quad \text{ for all } n > \nu_2.$$

Then

$$\frac{d(Su_n, Tu_n)}{2} = \frac{d(Su_n, Su_{n+1})}{2} < \frac{\beta}{4} < d(Su_n, SSu_n).$$

So, by (1) we obtain that

$$d(Tu_n, TSu_n) < \max\{d(Su_n, SSu_n), \frac{1}{2}(d(Su_n, Tu_n) + d(SSu_n, TSu_n))\}$$

$$d(Su_{n+1}, SSu_{n+1}) < \max\{d(Su_n, SSu_n), \frac{1}{2}(d(Su_n, Su_{n+1}) + d(SSu_n, SSu_{n+1}))\}$$

Since

$$\frac{1}{2}(d(Su_n, Su_{n+1}) + d(SSu_n, SSu_{n+1})) < \frac{\beta}{2} < d(Su_n, SSu_n)$$

we get

$$d(Su_{n+1}, SSu_{n+1}) < d(Su_n, SSu_n), \text{ for all } n > \nu_2.$$

This implies that $\{p(Su_n, SSu_{n+1})\}$ is strictly decreasing for large $n \in \mathbb{N}$ and

$$d(Su_n, SSu_n) > \beta \text{ for } n \geq \nu_2.$$

Then we can take $j \in \mathbb{N}$ such that $j \geq \nu_2$ and $d(Su_j, SSu_j) < \beta + \delta(\beta)$.

Then

$$d(Su_j, Tu_j) = d(Su_j, Su_{j+1}) < \frac{\beta}{2} < 2d(Su_j, SSu_j).$$

That is, $\frac{1}{2}d(Su_j, Tu_j) \leq d(Su_j, SSu_j)$
and

$$d(Su_{j+1}, SSu_{j+1}) = \max\{d(Su_j, SSu_j), \frac{1}{2}(d(Su_j, Tu_j) + d(SSu_j, TSu_j))\} < \beta + \delta(\beta). \tag{8}$$

Therefore, from (2) we have,

$$d(Tu_j, TSu_j) \leq \beta \tag{9}$$

Since, $\{T, S\}$ is compatible and $Su_n \to z$ and $Tu_n \to z$ as $n \to \infty$, thus for $\nu_3 \in \mathbb{N}$

$$d(STu_j, TSu_j) \leq \beta \text{ for all } j \geq \nu_3.$$

Choose $\nu_* = \max\{\nu_2, \nu_3\}$. Thus for all $j \geq \nu_*$, we get

$$\begin{aligned} d(Su_{j+1}, SSu_{j+1}) &= d(Tu_j, STu_j) \\ &\leq d(Tu_j, TSu_j) + d(STu_j, TSu_j) \\ &\leq 2\beta \end{aligned}$$

which contradicts (8). Thus we obtain that $Sz = z$. Let us prove $Tz = z$. If there exist $\nu \in \mathbb{N}$ such that $Su_\nu = Su_{\nu+1}$ then $Su_\nu = Tu_\nu$ and by construction of I we obtain $u_\nu = u_{\nu+1}$.
Hence $u_n = u_\nu$ for all $n \geq \nu$. Since $Su_n \to z$ we have $Su_n = z$ for $n \geq \nu$ and then

$$Tz = TSu_\nu = STu_\nu = SSu_{\nu+1} = Sz = z.$$

In the other case, we have

$$Su_n \neq Su_{n+1} \text{ for all } n \in \mathbb{N}.$$

So

$$Su_n \neq Tu_n \text{ for } n \in \mathbb{N}.$$

If

$$d(Su_n, Su_{n+1}) \geq 2d(Su_n, z)$$

and

$$d(Su_{n+1}, Su_{n+2}) \geq 2d(Su_{n+1}, z),$$

then we have by (3)

$$\begin{aligned} d(Su_n, Su_{n+1}) &\leq d(Su_n, z) + d(Su_{n+1}, z) \\ &\leq d(Su_n, z) + d(Su_{n+1}, z) \\ &\leq \frac{1}{2}(d(Su_n, Su_{n+1}) + d(Su_{n+1}, Su_{n+2})) \\ &< d(Su_n, Su_{n+1}) \end{aligned}$$

This is a contradiction. Therefore we have
either

$$d(Su_n, Su_{n+1}) < 2d(Su_n, z)$$

or

$$d(Su_{n+1}, Su_{n+2}) < 2d(Su_{n+1}, z) \text{ for } n \in \mathbb{N}.$$

Then from (1)
either

$$d(Tu_n, Tz) < \max\{d(Su_n, Sz), \frac{1}{2}(d(Su_n, Tu_n) + d(Sz, Tz))\}$$

or

$$d(Tu_{n+1}, Tz) < \max\{d(Su_{n+1}, Sz), \frac{1}{2}(d(Su_{n+1}, Tu_{n+1}) + d(Sz, Tz))\}$$

holds for $n \in \mathbb{N}$. Therefore, there exist a subsequence $\{n_j\}$ of $\{n\}$ such that

$$d(Tu_{n_j}, Tz) < \max\{d(Su_{n_j}, Sz), \frac{1}{2}(d(Su_{n_j}, Tu_{n_j}) + d(Sz, Tz))\}$$

holds for $j \in \mathbb{N}$.
Since $Tu_{n_j} = Su_{n_{j+1}}$ and $Su_n \to z$ we obtain that

$$\begin{aligned} d(z, Tz) &\leq \max\{d(z, Sz), \frac{1}{2}(d(z, z) + d(Sz, Tz))\} \\ &= \frac{1}{2}d(z, Tz) \quad [\text{since } z = Sz]. \end{aligned}$$

which implies that, $Tz = z$.
Hence in all cases, we have shown z is a common fixed point of S and T.
We suppose that y is another common fixed point of S and T. Since

$$\frac{1}{2}d(Sz, Tz) < d(z, y) = d(Sz, Sy)$$

we have by (1)

$$\begin{aligned} d(z, y) = d(Tz, Ty) &< \max\{d(Sz, Sy), \frac{1}{2}(d(Sz, Tz) + d(Sy, Ty))\} \\ &= d(Sz, Sy) \\ &= d(z, y) \end{aligned}$$

This is a contradiction. So the common fixed point is unique. □

Example 3.2. *Let* $X = \{0, 1, 2, 3, 4............\}$, *and*

$$d(x, y) = x + y + 4 \text{ when } x \neq y \text{ for all } x,\ y \in X;$$
$$d(x, y) = 0 \text{ when } x = y.$$

Therefore (X, d) *be a complete metric space.*
Define two functions $S,\ T$ *on* X *as follows:*

$$Sx = 2x$$

$$Tx = \begin{cases} 2[\frac{x}{20}], & if \quad 0 \leq x < 60; \\ 0, & if \quad 60 \leq x. \end{cases}$$

where $[x]$ is the greatest integer not greater than x. It is clear that $T0 = 0 = S0$, otherwise $Sx \neq Tx$ for all $x \in X$. We can define a mapping I on X by $Ix = [\frac{x}{20}]$ if $0 \leq x < 60$ and $Ix = 0$ if $x \geq 60$, where $[x]$ is the greatest integer not greater than x.

***Case I**.*

Let $0 \leq x < 60$ and $y \in X$.

$$\frac{1}{2}d(Sx, Tx) = \frac{1}{2}(2x + 2[\frac{x}{20}] + 4)$$

$$\frac{1}{2}d(Sx, Tx) = \begin{cases} x + 0 + 2 = x + 2, & if \quad 0 \leq x < 20; \\ x + 1 + 2 = x + 3, & if \quad 20 \leq x < 40; \\ x + 2 + 2 = x + 4, & if \quad 40 \leq x < 60. \end{cases}$$

Therefore,

$$\frac{1}{2}d(Sx, Tx) < d(Sx, Sy).$$

And

$$\begin{aligned} \max\{d(Sx, Sy), \frac{1}{2}(d(Sx, Tx)+d(Sy, Ty))\} &= \max\{d(Sx, Sy), \frac{1}{2}(d(Sx, Tx) + d(Sy, Ty))\} \\ &= \max\{2x + 2y + 4, \frac{1}{2}(2x + [\frac{x}{20}] + 4 + 2y + [\frac{y}{20}] + 4)\} \\ &= 2x + 2y + 4 \\ &= d(Sx, Sy). \end{aligned}$$

For $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that,
$\max\{d(Sx, Sy), \frac{1}{2}(d(Sx, Tx) + d(Sy, Ty))\} < \varepsilon + \delta(\varepsilon)$ implies $d(Tx, Ty) = 0 < \varepsilon$.
Hence the result is true for $0 \leq x < 60$ and $y \in X$.
***Case II.** Let $60 \leq x$ and $y \in X$.*
In this case $Tx = 0$, the result is obvious.
0 is the unique common fixed point of S and T.

Result in partial metric space.

Lemma 3.3. *Let (X, p) be a partial metric space, T a self map on X, d the constructed metric in Proposition 2.5 and $x, y \in X$. Then*
$\max\{d(Sx, Sy), \frac{1}{2}(d(Sx, Tx) + d(Sy, Ty))\} = \max\{p(Sx, Sy), \frac{1}{2}(p(Sx, Tx) + p(Sy, Ty))\}$ *for all $x, y \in X$ with $x \neq y$.*

Proof. The proof of the lemma is almost identical with that of Lemma 2.2 in [10]. We do not give the details of proof here. Instead, we refer it to [10]. □

Theorem 3.4. *Let (X, p) be a complete partial metric space. Let S be a continuous mappings on X, and T be another mapping on X such that $\{T, S\}$ is partial compatible and $T(X) \subset S(X)$.*

Also let for all $x, y \in X$ and for any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that

$$(i)\frac{1}{2}p(Sx,Tx) < p(Sx,Sy) \Rightarrow p(Tx,Ty) < \max\{p(Sx,Sy), \frac{1}{2}(p(Sx,Tx)+p(Sy,Ty))\};$$

$$(ii)\frac{1}{2}p(Sx,Tx) < p(Sx,Sy) \text{ and } \max\{p(Sx,Sy), \frac{1}{2}(p(Sx,Tx)+p(Sy,Ty))\} < \varepsilon+\delta(\varepsilon)$$
$$\Rightarrow p(Tx,Ty) \leq \varepsilon.$$

Then there exist a unique common fixed point of S and T.

Proof. By using Proposition 2.5, $(X,\ d)$ is a complete metric space, where d is the constructed metric. By Lemma 3.3, we have

$$\max\{p(Sx,Sy), \frac{1}{2}(d(Sx,Tx)+d(Sy,Ty))\} = \max\{p(Sx,Sy), \frac{1}{2}(p(Sx,Tx)+p(Sy,Ty))\}.$$

Therefore, for all $x, y \in X$ with $x \neq y$ we have,

$$\frac{1}{2}p(Sx,Tx) = \frac{1}{2}d(Sx,Tx) < d(Sx,Sy) = p(Sx,Sy).$$

For any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that

$$\frac{1}{2}p(Sx,Tx) = \frac{1}{2}d(Sx,Tx) < d(Sx,Sy) = p(Sx,Sy)$$

and

$$\max\{p(Sx,Sy), \frac{1}{2}(p(Sx,Tx)+p(Sy,Ty))\} = \max\{d(Sx,Sy), \frac{1}{2}(d(Sx,Tx)+d(Sy,Ty))\}$$
$$< \varepsilon+\delta(\varepsilon)$$

$$\Rightarrow d(Tx,Ty) = p(Tx,Ty) \leq \varepsilon.$$

Then, by using Theorem3.1, S, T have unique common fixed point. □

Example 3.5. *Let*

$$X = [0,2] \;\; p(x,y) = \max\{x,y\} \text{ for all } x \in X.$$

Therefore (X, p) be a complete partial metric space.
Define two functions $S,\ T$ as follows:

$$Sx = \begin{cases} 2x, & \text{if} \quad x \in [0,1]; \\ 3-x, & \text{if} \quad x \in [1,2]. \end{cases}$$

$$Tx = \begin{cases} 0, & \text{if} \quad 0 \leq x < 1; \\ \frac{1}{x}, & \text{if} \quad 1 \leq x \leq 2. \end{cases}$$

It is clear that $T0 = 0 = S0$, *otherwise* $Sx \neq Tx$ *for all* $x \in X$.
Case I.
Let $0 \leq x < 1$. *Then*

$$\frac{1}{2}p(Sx, Tx) = \frac{1}{2}p(2x, 0) = x < \max\{2x, 2y\} = p(Sx, Sy).$$

Therefore,

$$\begin{aligned}\max\{p(Sx, Sy), \frac{1}{2}(p(Sx, Tx) + p(Sy, Ty))\} &= \max\{p(2x, 2y), \frac{1}{2}(p(2x, 0) + p(2y, 0))\} \\ &= \max\{\max\{2x, 2y\}, \frac{1}{2}(2x + 2y)\} \\ &= \max\{2x, 2y\}. \end{aligned} \tag{10}$$

For $\varepsilon > 0$, *there exist* $\delta(\varepsilon) > 0$ *such that*
$\max\{p(Sx, Sy), \frac{1}{2}(p(Sx, Tx) + p(Sy, Ty))\} < \varepsilon + \delta(\varepsilon)$ *implies* $p(Tx, Ty) = 0 < \varepsilon$.
Hence the result is true for $0 \leq x < 1$.
Case II. *Let* $1 \leq x \leq 2$.
$\frac{1}{2}p(Sx, Tx) = \frac{1}{2}(3 - x) < \max\{3 - x, 3 - y\} = p(Sx, Sy)$.
Now,

$$\begin{aligned}\max\{p(Sx, Sy), \frac{1}{2}(p(Sx, Tx) + p(Sy, Ty))\} &= \max\{\max\{3 - x, 3 - y\}, \frac{1}{2}(3 - x + 3 - y)\} \\ &= \max\{3 - x, 3 - y\}. \end{aligned} \tag{11}$$

We also have $p(Tx, Ty) = \max\{\frac{1}{x}, \frac{1}{y}\}$. *(12)*
Now for the given $\varepsilon > 0$, *there exist* $\delta(\varepsilon) > 0$ *such that*

$$\max\{p(Sx, Sy), \frac{1}{2}(p(Sx, Tx) + p(Sy, Ty))\} < \varepsilon + \delta(\varepsilon).$$

Using (11), (12) and the above inequality implies $p(Tx, Ty) < \varepsilon$.
Hence the result satisfied for $1 \leq x \leq 2$.
0 *is the unique common fixed point of* S *and* T.

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